CHERN NUMBERS AND THE INDICES OF SOME ELLIPTIC DIFFERENTIAL OPERATORS

PING LI

Abstract. Libgober and Wood proved that the Chern number c_1c_{n-1} of a n-dimensional compact complex manifold can be determined by its Hirzebruch χ_y-genus. Inspired by the idea of their proof, we show that, for compact, spin, almost-complex manifolds, more Chern numbers can be determined by the indices of some twisted Dirac and signature operators. As a byproduct, we get a divisibility result of certain characteristic number for such manifolds. Using our method, we give a direct proof of Libgober-Wood's result, which was originally proved by induction.

1. Introduction and main results

Suppose (M, J) is a compact, almost-complex $2n$-manifold with a given almost complex structure J. This J makes the tangent bundle of M into a n-dimensional complex vector bundle T_M. Let $c_i(M, J) \in H^{2i}(M; \mathbb{Z})$ be the i-th Chern class of T_M. Suppose we have a formal factorization of the total Chern class as follows:

$$1 + c_1(M, J) + \cdots + c_n(M, J) = \prod_{i=1}^{n}(1 + x_i),$$

i.e., $x_1, \cdots, x_n$ are formal Chern roots of T_M. The Hirzebruch χ_y-genus of (M, J), $\chi_y(M, J)$, is defined by

$$\chi_y(M, J) = [\prod_{i=1}^{n} \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}][M].$$

Here $[M]$ is the fundamental class of the orientation of M induced by J and y is an indeterminate. If J is specified, we simply denote $\chi_y(M, J)$ by $\chi_y(M)$.

When the almost complex structure J is integrable, i.e., M is a (complex) n-dimensional compact complex manifold, $\chi_y(M)$ can be obtained by

$$\chi^p(M) = \sum_{q=0}^{n}(-1)^q h^{p,q}(M),\ \chi_y(M) = \sum_{p=0}^{n}\chi^p(M) \cdot y^p,$$

where $h^{p,q}(\cdot)$ is the Hodge number of type (p, q). This is given by the Hirzebruch-Riemann-Roch Theorem. It was first proved by Hirzebruch ([2]) for projective manifolds, and in the general case by Atiyah and Singer ([1]).

The formula

$$\sum_{p=0}^{n}\chi^p(M) \cdot y^p = [\prod_{i=1}^{n} \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}][M] \tag{1.1}$$

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implies that $\chi^p(M)$ (the index of the Dolbeault complex) can be expressed as a rationally linear combination of some Chern numbers of M. Conversely, we can address the following question.

Question 1.1. For a n-dimensional compact complex manifold M, given a partition $\lambda = \lambda_1\lambda_2\cdots\lambda_l$ of weight n, whether the corresponding Chern number $c_{\lambda_1}c_{\lambda_2}\cdots c_{\lambda_l}[M]$ can be determined by $\chi^p(M)$, or more generally by the indices of some other elliptic differential operators?

For the simplest case $c_n[M]$, the answer is affirmative and well-known ([2], Theorem 15.8.1):

$$c_n[M] = \chi_y(M)|_{y=-1} = \sum_{p=0}^{n}(-1)^p\chi^p(M).$$

The next-to-simplest case is the Chern number $c_1c_{n-1}[M]$. The answer for this Chern number is also affirmative, which was proved by Libgober and Wood ([5], p.141-143):

$$\sum_{p=2}^{n}(-1)^p\binom{p}{2}\chi^p(M) = \frac{n(3n-5)}{24}c_n[M] + \frac{1}{12}c_1c_{n-1}[M]. \tag{1.2}$$

The idea of their proof is quite enlightening: expanding both sides of (1.1) at $y = -1$ and comparing the coefficients of the term $(y+1)^2$, then they got (1.2).

Inspired by this idea, in this paper we will consider the twisted Dirac operators and signature operators on compact, *spin*, *almost-complex* manifolds and show that the Chern numbers c_n, c_1c_{n-1}, $c_1^2c_{n-2}$ and c_2c_{n-2} of such manifolds can be determined by the indices of these operators.

Remark 1.2. It is worth pointing out that (1.2) was also obtained later in ([7], p.144) by Salamon and he majorly applied this result to hyper-Kähler manifolds.

Before stating our main result, we need to fix some definitions and symbols. Let M be a compact, *almost-complex* $2n$-manifold. We still use $x_1,\cdots,x_n$ to denote the corresponding formal Chern roots of the n-dimensional complex vector bundle T_M. Suppose E is a complex vector bundle over M. Set

$$\hat{A}(M,E) := [\mathrm{ch}(E)\cdot\prod_{i=1}^{n}\frac{x_i/2}{\sinh(x_i/2)}][M],$$

and

$$L(M,E) := [\mathrm{ch}(E)\cdot\prod_{i=1}^{n}\frac{x_i(1+e^{-x_i})}{1-e^{-x_i}}][M],$$

where $\mathrm{ch}(E)$ is the Chern character of E. The celebrated Atiyah-Singer index theorem (cf. [4], p.74 and p.81) states that $L(M,E)$ is the index of the signature operator twisted by E and when M is *spin*, $\hat{A}(M,E)$ is the index of the Dirac operator twisted by E.

Definition 1.3.

$$A_y(M) := \sum_{p=0}^{n}\hat{A}(M,\Lambda^p(T_M^*))\cdot y^p,$$

$$L_y(M) := \sum_{p=0}^{n}L(M,\Lambda^p(T_M^*))\cdot y^p,$$

where $\Lambda^p(T_M^*)$ denotes the p-th exterior power of the dual of T_M.

Our main result is the following

Theorem 1.4. *If M is a compact, almost-complex manifold, then we have*

(1)
$$\sum_{p=0}^{n}(-1)^p\hat{A}(M,\Lambda^p(T_M^*))=c_n[M],$$

$$\sum_{p=1}^{n}(-1)^p\cdot p\cdot\hat{A}(M,\Lambda^p(T_M^*))=\frac{1}{2}\{nc_n[M]+c_1c_{n-1}[M]\},$$

$$\sum_{p=2}^{n}(-1)^p\binom{p}{2}\hat{A}(M,\Lambda^p(T_M^*))=\{\frac{n(3n-5)}{24}c_n+\frac{3n-2}{12}c_1c_{n-1}+\frac{1}{8}c_1^2c_{n-2}\}[M];$$

(2)
$$\sum_{p=0}^{n}(-1)^pL(M,\Lambda^p(T_M^*))=2^nc_n[M],$$

$$\sum_{p=1}^{n}(-1)^p\cdot p\cdot L(M,\Lambda^p(T_M^*))=2^{n-1}\{nc_n[M]+c_1c_{n-1}[M]\},$$

$$\sum_{p=2}^{n}(-1)^p\binom{p}{2}L(M,\Lambda^p(T_M^*))=2^{n-2}\{\frac{n(3n-5)}{6}c_n+\frac{3n-2}{3}c_1c_{n-1}+c_1^2c_{n-2}-c_2c_{n-2}\}[M].$$

Corollary 1.5. (1) The Chern numbers $c_n[M]$, $c_1c_{n-1}[M]$ and $c_1^2c_{n-2}[M]$ can be determined by $A_y(M)$.

(2) The characteristic numbers $c_n[M]$, $c_1c_{n-1}[M]$ and $c_1^2c_{n-2}[M]-c_2c_{n-2}[M]$ can be determined by $L_y(M)$.

(3) When M is *spin*, the Chern numbers $c_n[M]$, $c_1c_{n-1}[M]$, $c_1^2c_{n-2}[M]$ and $c_2c_{n-2}[M]$ can be expressed by using linear combinations of the indices of some twisted Dirac and signature operators.

As remarked in page 143 of [5], it was shown by Milnor (cf. [3]) that every complex cobordism class contains a non-singular algebraic variety. Milnor ([6]) also showed that two almost-complex manifolds are complex cobordant if and only if they have the same Chern numbers. Hence Libgober and Wood's result implies that the characteristic number

$$\frac{n(3n-5)}{24}c_n[N]+\frac{1}{12}c_1c_{n-1}[N]$$

is always an integer for any compact, *almost-complex* $2n$-manifold N.

Note that the right-hand side of the third equality in Theorem 1.4 is

$$\{\frac{n(3n-5)}{24}c_n[M]+\frac{1}{12}c_1c_{n-1}[M]\}+\frac{1}{8}\{2(n-1)c_1c_{n-1}[M]+c_1^2c_{n-2}[M]\}.$$

Hence we get the following divisibility result.

Corollary 1.6. For a compact, *spin*, almost-complex manifold M, the integer $2(n-1)c_1c_{n-1}[M]+c_1^2c_{n-2}[M]$ is divisible by 8.

Example 1.7. The total Chern class of the complex projective space $\mathbb{C}P^n$ is given by $c(\mathbb{C}P^n)=(1+g)^{n+1}$, where g is the standard generator of $H^2(\mathbb{C}P^n;\mathbb{Z})\cong\mathbb{Z}$. $\mathbb{C}P^n$ is spin if and only if n is odd as $c_1(\mathbb{C}P^n)=(n+1)g$. Suppose $n=2k+1$. Then

$$2(n-1)c_1c_{n-1}[\mathbb{C}P^n]+c_1^2c_{n-2}[\mathbb{C}P^n]=8(k+1)^2[k(2k+1)+\frac{1}{3}k(k+1)(2k+1)].$$

While it is easy to check that $\mathbb{C}P^4$ does not satisfy this divisibility result.

2. Proof of the main result

Lemma 2.1.

$$A_y(M)=\{\prod_{i=1}^{n}[\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot e^{-\frac{x_i(1+y)}{2}}]\}[M],$$

$$L_y(M)=\{\prod_{i=1}^{n}[\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{-x_i(1+y)})]\}[M].$$

Proof. From $c(T_M)=\prod_{i=1}^n(1+x_i)$ we have (see, for example, [4], p.11)

$$c(\Lambda^p(T_M^*))=\prod_{1\leq i_1<\cdots<i_p\leq n}[1-(x_{i_1}+\cdots+x_{i_p})].$$

Hence

$$ch(\Lambda^p(T_M^*))y^p=\sum_{1\leq i_1<\cdots<i_p\leq n}e^{-(x_{i_1}+\cdots+x_{i_p})}y^p=\sum_{1\leq i_1<\cdots<i_p\leq n}(\prod_{j=1}^{p}ye^{-x_{i_j}}).$$

Therefore we have

$$\sum_{p=0}^{n}ch(\Lambda^p(T_M^*))y^p=\sum_{p=0}^{n}[\sum_{1\leq i_1<\cdots<i_p\leq n}(\prod_{j=1}^{p}ye^{-x_{i_j}})]=\prod_{i=1}^{n}(1+ye^{-x_i}).$$

So

$$\begin{aligned}A_y(M)&=\sum_{p=0}^{n}\hat{A}(M,\Lambda^p(T_M^*))\cdot y^p\\&=\{[\sum_{p=0}^{n}ch(\Lambda^p(T_M^*))y^p]\cdot\prod_{i=1}^{n}\frac{x_i/2}{\sinh(x_i/2)}\}[M]\\&=\{\prod_{i=1}^{n}[(1+ye^{-x_i})\cdot\frac{x_i/2}{\sinh(x_i/2)}]\}[M]\\&=\{\prod_{i=1}^{n}[\frac{x_i(1+ye^{-x_i})}{1-e^{-x_i}}\cdot e^{-\frac{x_i}{2}}]\}[M].\end{aligned}\tag{2.1}$$

Since for the evaluation only the homogeneous component of degree n in the x_i enters, then we obtain an additional factor $(1+y)^n$ if we replace x_i by $x_i(1+y)$ in (2.1). We therefore obtain:

$$
\begin{aligned}
A_y(M) &= \{\frac{1}{(1+y)^n}\prod_{i=1}^{n}[\frac{x_i(1+y)(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot e^{-\frac{x_i(1+y)}{2}}]\}[M] \\
&= \{\prod_{i=1}^{n}[\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot e^{-\frac{x_i(1+y)}{2}}]\}[M].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_y(M) &= \{\prod_{i=1}^{n}[(1+ye^{-x_i})\cdot\frac{x_i(1+e^{-x_i})}{1-e^{-x_i}}]\}[M] \\
&= \{\frac{1}{(1+y)^n}\prod_{i=1}^{n}[\frac{x_i(1+y)(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{-x_i(1+y)})]\}[M] \\
&= \{\prod_{i=1}^{n}[\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{x_i(1+y)})]\}[M].
\end{aligned}
$$

□

Lemma 2.2. *Set $z=1+y$. We have*

$$
A_y(M) = \{\prod_{i=1}^{n}[(1+x_i)-(x_i+\frac{1}{2}x_i^2)z+(\frac{11}{24}x_i^2+\frac{1}{8}x_i^3)z^2+\cdots]\}[M],
$$

$$
L_y(M) = \{\prod_{i=1}^{n}[2(1+x_i)-(2x_i+x_i^2)z+(\frac{7}{6}x_i^2+\frac{1}{2}x_i^3)z^2+\cdots]\}[M].
$$

Proof.

$$
\begin{aligned}
\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}} &= -x_iy+\frac{x_i(1+y)}{1-e^{-x_i(1+y)}} = -x_i(z-1)+\frac{x_iz}{1-e^{-x_iz}} \\
&= -x_i(z-1)+(1+\frac{1}{2}x_iz+\frac{1}{12}x_i^2z^2+\cdots) \\
&= (1+x_i)-\frac{1}{2}x_iz+\frac{1}{12}x_i^2z^2+\cdots.
\end{aligned}
$$

So we have

$$
\begin{aligned}
A_y(M) &= \{\prod_{i=1}^{n}[\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot e^{-\frac{x_i(1+y)}{2}}]\}[M] \\
&= \{\prod_{i=1}^{n}[(1+x_i)-\frac{1}{2}x_iz+\frac{1}{12}x_i^2z^2+\cdots][1-\frac{1}{2}x_iz+\frac{1}{8}x_i^2z^2+\cdots]\}[M] \\
&= \{\prod_{i=1}^{n}[(1+x_i)-(x_i+\frac{1}{2}x_i^2)z+(\frac{11}{24}x_i^2+\frac{1}{8}x_i^3)z^2+\cdots]\}[M].
\end{aligned}
$$

Similarly,

$$\begin{aligned}L_y(M) &= \{\prod_{i=1}^{n}[\frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}\cdot(1+e^{-x_i(1+y)})]\}[M]\\ &= \{\prod_{i=1}^{n}[(1+x_i)-\frac{1}{2}x_iz+\frac{1}{12}x_i^2z^2+\cdots][2-x_iz+\frac{1}{2}x_i^2z^2+\cdots]\}[M]\\ &= \{\prod_{i=1}^{n}[2(1+x_i)-(2x_i+x_i^2)z+(\frac{7}{6}x_i^2+\frac{1}{2}x_i^3)z^2+\cdots]\}[M].\end{aligned}$$

□

Let $f(x_1,\cdots,x_n)$ be a symmetric polynomial in $x_1,\cdots,x_n$. Then $f(x_1,\cdots,x_n)$ can be expressed in terms of $c_1,\cdots,c_n$ in a unique way. We use $h(f(x_1,\cdots,x_n))$ to denote the homogeneous component of degree n in $f(x_1,\cdots,x_n)$. For instance, when $n=3$,

$$\begin{aligned}&h(x_1+x_2+x_3+x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2)\\ &=x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2\\ &=(x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)-3x_1x_2x_3\\ &=c_1c_2-3c_3.\end{aligned}$$

The following lemma is a crucial technical ingredient in the proof of our main result.

Lemma 2.3.

(1) $h_1 := h\{\sum_{i=1}^{n}[x_i\prod_{j\neq i}(1+x_j)]\} = nc_n$,
(2) $h_{11} := h\{\sum_{1\leq i<j\leq n}[x_ix_j\prod_{k\neq i,j}(1+x_k)]\} = \frac{n(n-1)}{2}c_n$,
(3) $h_2 := h\{\sum_{i=1}^{n}[x_i^2\prod_{j\neq i}(1+x_j)]\} = -nc_n+c_1c_{n-1}$,
(4) $h_{12} := h\{\sum_{1\leq i<j\leq n}[(x_i^2x_j+x_ix_j^2)\prod_{k\neq i,j}(1+x_k)]\} = (n-2)(-nc_n+c_1c_{n-1})$,
(5) $h_{22} := h\{\sum_{1\leq i<j\leq n}[x_i^2x_j^2\prod_{k\neq i,j}(1+x_k)]\} = \frac{n(n-3)}{2}c_n-(n-2)c_1c_{n-1}+c_2c_{n-2}$,
(6) $h_3 := h\{\sum_{i=1}^{n}[x_i^3\prod_{j\neq i}(1+x_j)]\} = nc_n-c_1c_{n-1}+c_1^2c_{n-2}-2c_2c_{n-2}$.

Now we can complete the proof of Theorem 1.4 and postpone the proof of Lemma 2.3 to the end of this section.

Proof. From Lemma 2.2, the constant coefficient of $A_y(M)$ is $[\prod_{i=1}^{n}(1+x_i)][M]=c_n[M]$.

The coefficient of z is

$$\{\sum_{i=1}^{n}[-(x_i+\frac{1}{2}x_i^2)\prod_{j\neq i}(1+x_j)]\}[M]=(-h_1-\frac{1}{2}h_2)[M]=-\frac{1}{2}\{nc_n[M]+c_1c_{n-1}[M]\}.$$

The coefficient of z^2 is

$$\begin{aligned}&\{\sum_{i=1}^{n}[(\frac{11}{24}x_i^2+\frac{1}{8}x_i^3)\prod_{j\neq i}(1+x_j)]+\sum_{1\leq i<j\leq n}[(x_i+\frac{1}{2}x_i^2)(x_j+\frac{1}{2}x_j^2)\prod_{k\neq i,j}(1+x_k)]\}[M]\\ &=(\frac{11}{24}h_2+\frac{1}{8}h_3+h_{11}+\frac{1}{2}h_{12}+\frac{1}{4}h_{22})[M]\\ &=\{\frac{n(3n-5)}{24}c_n+\frac{3n-2}{12}c_1c_{n-1}+\frac{1}{8}c_1^2c_{n-2}\}[M].\end{aligned}$$

Similarly, for $L_y(M)$, the constant coefficient is $[2^n\prod_{i=1}^{n}(1+x_i)][M]=2^nc_n[M]$.

The coefficient of z is

$$\{\sum_{i=1}^{n}[-(2x_i+x_i^2)\prod_{j\neq i}2(1+x_j)]\}[M]=(-2^n h_1-2^{n-1}h_2)[M]=-2^{n-1}\{nc_n[M]+c_1c_{n-1}[M]\}.$$

The coefficient of z^2 is

$$\begin{aligned}
&\{\sum_{i=1}^{n}[(\frac{7}{6}x_i^2+\frac{1}{2}x_i^3)\prod_{j\neq i}2(1+x_j)]+\sum_{1\leq i<j\leq n}[(2x_i+x_i^2)(2x_j+x_j^2)\prod_{k\neq i,j}2(1+x_k)]\}[M]\\
&=(\frac{7\cdot 2^{n-2}}{3}h_2+2^{n-2}h_3+2^n h_{11}+2^{n-1}h_{12}+2^{n-2}h_{22})[M]\\
&=2^{n-2}\{\frac{n(3n-5)}{6}c_n+\frac{3n-2}{3}c_1c_{n-1}+c_1^2c_{n-2}-c_2c_{n-2}\}[M].
\end{aligned}$$

□

It suffices to show Lemma 2.3.

Proof. (1) and (2) are quite obvious. In the following proof, $\hat{x_i}$ means deleting x_i.

For (3),

$$\begin{aligned}
h_2&=\sum_{i=1}^{n}\{h[x_i^2\prod_{j\neq i}(1+x_j)]\}\\
&=\sum_{i=1}^{n}(x_i\sum_{j\neq i}x_1\cdots\hat{x_j}\cdots x_n)\\
&=\sum_{i=1}^{n}(x_ic_{n-1}-c_n)\\
&=-nc_n+c_1c_{n-1}.
\end{aligned}$$

For (4),

$$\begin{aligned}
h_{12}&=\sum_{1\leq i<j\leq n}\{h[(x_i^2x_j+x_ix_j^2)\prod_{k\neq i,j}(1+x_k)]\}\\
&=\sum_{1\leq i<j\leq n}[(x_i+x_j)\sum_{k\neq i,j}x_1\cdots\hat{x_k}\cdots x_n]\\
&=(n-2)\sum_{i=1}^{n}(x_i\sum_{k\neq i}x_1\cdots\hat{x_k}\cdots x_n)\\
&=(n-2)h_2\\
&=(n-2)(-nc_n+c_1c_{n-1}).
\end{aligned}$$

For (5),

$$\begin{aligned}
c_2c_{n-2} &= (\sum_{1\leq i<j\leq n} x_ix_j)(\sum_{1\leq k<l\leq n} x_1\cdots\hat{x_k}\cdots\hat{x_l}\cdots x_n)\\
&= \sum_{1\leq i<j\leq n}(x_ix_j \sum_{1\leq k<l\leq n} x_1\cdots\hat{x_k}\cdots\hat{x_l}\cdots x_n)\\
&= \sum_{1\leq i<j\leq n}[x_1x_2\cdots x_n + (x_i^2x_j + x_ix_j^2)\sum_{k\neq i,j} x_1\cdots\hat{x_k}\cdots\hat{x_i}\cdots\hat{x_j}\cdots x_n +\\
&\quad x_i^2x_j^2 \sum_{1\leq k<l\leq n; k\neq i,j; l\neq i,j.} x_1\cdots\hat{x_k}\cdots\hat{x_l}\cdots\hat{x_i}\cdots\hat{x_j}\cdots x_n]\\
&= \frac{n(n-1)}{2}c_n + h_{12} + h_{22}.
\end{aligned}$$

Therefore,

$$h_{22} = c_2c_{n-2} - \frac{n(n-1)}{2}c_n - h_{12} = \frac{n(n-3)}{2}c_n - (n-2)c_1c_{n-1} + c_2c_{n-2}.$$

For (6),

$$\begin{aligned}
(c_1^2 - 2c_2)c_{n-2} &= (\sum_{i=1}^n x_i^2)(\sum_{1\leq j<k\leq n} x_1\cdots\hat{x_j}\cdots\hat{x_k}\cdots x_n)\\
&= \sum_{i=1}^n(x_i^2 \sum_{1\leq j<k\leq n} x_1\cdots\hat{x_j}\cdots\hat{x_k}\cdots x_n)\\
&= \sum_{i=1}^n[(x_i^3 \sum_{1\leq j<k\leq n; j\neq i; k\neq i.} x_1\cdots\hat{x_j}\cdots\hat{x_k}\cdots\hat{x_i}\cdots x_n) + (x_i^2\sum_{k\neq i} x_1\cdots\hat{x_k}\cdots\hat{x_i}\cdots x_n)]\\
&= h_3 + h_2.
\end{aligned}$$

Hence

$$h_3 = (c_1^2 - 2c_2)c_{n-2} - h_2 = nc_n - c_1c_{n-1} + c_1^2c_{n-2} - 2c_2c_{n-2}.$$

□

3. Concluding remarks

Libgober and Wood's proof of (1.2) is by induction ([5], p.142, Lemma 2.2). Here, using our method, we can give a quite direct proof. We have shown that

$$\begin{aligned}
\chi_y(M) &= [\prod_{i=1}^n \frac{x_i(1+ye^{-x_i(1+y)})}{1-e^{-x_i(1+y)}}][M]\\
&= \{\prod_{i=1}^n[(1+x_i) - \frac{1}{2}x_iz + \frac{1}{12}x_i^2z^2 + \cdots]\}[M]
\end{aligned}$$

The coefficient of z^2 is

$$\begin{aligned}
&\{\sum_{i=1}^{n}[\frac{1}{12}x_i^2\prod_{j\neq i}(1+x_j)]+\sum_{1\leq i<j\leq n}[\frac{1}{4}x_ix_j\prod_{k\neq i,j}(1+x_k)]\}[M]\\
&=(\frac{1}{12}h_2+\frac{1}{4}h_{11})[M]\\
&=\frac{n(3n-5)}{24}c_n[M]+\frac{1}{12}c_1c_{n-1}[M].
\end{aligned}$$

It is natural to ask what the coefficients are for higher order terms $(y+1)^p$ $(p\geq 3)$. Unfortunately the coefficients become very complicated for such terms. Libgober and Wood have given a detailed remark for higher order terms' coefficients of $\chi_y(M)$ ([5], p.144-145). Note that the expression of $A_y(M)$ (resp. $L_y(M)$) has an additional factor $e^{-\frac{x_i(1+y)}{2}}$ (resp. $1+e^{x_i(1+y)}$) than that of $\chi_y(M)$. Hence we cannot expect that there are *explicit* expressions of higher order terms' coefficients like Theorem 1.4.

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Department of Mathematics, Tongji University, Shanghai 200092, China

E-mail address: pingli@tongji.edu.cn